# The surface tangent paradox and the difference vector quotient of a secant plane 

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#### Abstract

If a one-variable function is sufficiently smooth, then the limit position of secant lines its graph is a tangent line. By analogy, one would expect that the limit position of secant planes of a two-variable smooth function is a plane tangent to its graph. Amazingly, this is not necessarily true, even when the function is a simple polynomial. Despite this paradox, we show that some analogies with the one-variable case still hold in the multi-variable context, provided we use a particular vector product: the Clifford's geometric one.


## 1 Introduction.

The classical Schwarz surface area paradox $\sqrt{1}$ provides a counterexample concerning the definition of the area of a curved surface. It is a divergence phenomenon, that is seldom presented, even at the end of an Advanced Calculus course ${ }^{2}$. Here, we show that a local version of this paradox deals with the very definition of the tangent to the graph of a smooth function. For simplicity, we only consider functions of one and two real variables.

Let us first recall the link between lines secant and tangent the graph of a one-variable smooth function $g: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$. If you consider the line passing through two distinct points $(a, g(a)),(b, g(b)) \in I \times \mathbb{R}$ of the graph of $g$, then this secant line becomes the line tangent the graph of $g$ at point $\left(x_{0}, g\left(x_{0}\right)\right)$, as the distinct points $a$ and $b$ converge towards $x_{0}$, whatever is the way they do it, provided $g$ is sufficiently smooth (a $C^{1}$-function, let's say). Analytically, this corresponds to the existence of the strong derivative ${ }^{3}$ of g at point $x_{0}$, which coincides with the classical derivative $g^{\prime}\left(x_{0}\right)$, provided the strong derivative

[^0]exists (as it happens for a $C^{1}$-function). More precisely,
\[

$$
\begin{equation*}
\lim _{\substack{a, b \rightarrow x_{0} \\ a \neq b}} \frac{g(b)-g(a)}{b-a}=g^{\prime}\left(x_{0}\right) \tag{1}
\end{equation*}
$$

\]

By analogy, one would expect that, for a two-variable smooth function, the plane tangent to its graph at a fixed point would be the limit position of the secant plane passing through three non-collinear points, as these three non-collinear points converge to that fixed point. Paradoxically, this is not the case.

### 1.1 Counterexample.

Let us consider as two-variable smooth function $f$ the polynomial $f(\mathbf{v})=\mathbf{v} \cdot \mathbf{v}=$ $x^{2}+y^{2}$, when $\mathbf{v}=x \mathbf{e}_{1}+y \mathbf{e}_{2}$, and $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ is an orthonormal basis of the twodimensional vector 4 Euclidean space $\mathbb{E}_{2}$, with scalar product $\mathbf{u} \cdot \mathbf{w}$, with $\mathbf{u}, \mathbf{w}$ vectors of $\mathbb{E}_{2}$. So, $\mathbf{e}_{1} \cdot \mathbf{e}_{2}=0$, and $\mathbf{e}_{1} \cdot \mathbf{e}_{1}=\mathbf{e}_{2} \cdot \mathbf{e}_{2}=1$. If $\mathbf{a}=\mathbf{0}, \mathbf{b}=-\delta \mathbf{e}_{1}+\eta \mathbf{e}_{2}$, and $\mathbf{c}=\delta \mathbf{e}_{1}+\eta \mathbf{e}_{2}$, then the Cartesian equation of the plane passing through points $(\mathbf{a}, f(\mathbf{a})),(\mathbf{b}, f(\mathbf{b}))$, and $(\mathbf{c}, f(\mathbf{c}))$ can be represented, in the $(x, y, z)$ coordinates, by the relation

$$
\begin{equation*}
\eta z=\left(\delta^{2}+\eta^{2}\right) y \tag{2}
\end{equation*}
$$

As the limit of $\frac{\delta^{2}+\eta^{2}}{\eta}$, as $\delta, \eta \rightarrow 0$, does not exist, then the secant plane (2) does not necessarily converge to the plane $z=0$, which is tangent to the graph of $f$ at point $\mathbf{0}$.

This counterexample illustrates the tangent paradox: a plane secant the graph of a smooth two-variable function $f$ at points $(\mathbf{a}, f(\mathbf{a})),(\mathbf{b}, f(\mathbf{b}))$, and $(\mathbf{c}, f(\mathbf{c}))$ can assume limit positions that depend on the way the three noncollinear points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ converge to $\mathbf{x}_{0}$. For example, if you consider $f, \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ as in the counterexample,

- if $\eta=\delta \rightarrow 0$, then the limit plane would be $z=0$, the tangent plane;
- if $\eta=\delta^{2} \rightarrow 0$, then the limit plane would be $z=y$, which is not the tangent plane;
- if $\eta=\delta^{3} \rightarrow 0$, then the limit plane would be $y=0$, which is even orthogonal to the tangent plane!

So, contrary to the one-variable case, we cannot simply say that the tangent plane to a smooth two-variable function is the limit position of its secant planes.

[^1]
### 1.2 In search for analogies.

Despite such paradox, one can ask if any analogy with the one-variable case still holds for two-variable smooth functions.
In the $(x, z)$-plane, the Cartesian equation of the line passing through $(a, g(a))$, $(b, g(b))($ with $a \neq b)$ is

$$
\begin{equation*}
z=g(a)+\frac{g(b)-g(a)}{b-a}(x-a) \tag{3}
\end{equation*}
$$

The existence of the strong derivative (11), and the continuity of $g$ at $x_{0}$ imply that, as $a, b \rightarrow x_{0}$, relation (3) becomes relation

$$
z=g\left(x_{0}\right)+g^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)
$$

which represents the line tangent the graph of $g$ at point $\left(x_{0}, g\left(x_{0}\right)\right)$. The relation representing the plane tangent the graph of the two-variable function $f$ : $\Omega \subseteq \mathbb{E}_{2} \rightarrow \mathbb{R}$ at point $\left(\mathbf{v}_{0}, f\left(\mathbf{v}_{0}\right)\right)$ can be written in the $(\mathbf{v}, z)$-space $\mathbb{E}_{2} \times \mathbb{R}$ as

$$
z=f\left(\mathbf{v}_{0}\right)+\nabla f\left(\mathbf{v}_{0}\right) \cdot\left(\mathbf{v}-\mathbf{v}_{0}\right) .
$$

So, we wonder if, in analogy with (3), there exists a relation

$$
\begin{equation*}
z=f(\mathbf{a})+\mathbf{q} \cdot(\mathbf{v}-\mathbf{a}) \tag{4}
\end{equation*}
$$

representing the plane passing through $(\mathbf{a}, f(\mathbf{a})),(\mathbf{b}, f(\mathbf{b}))$, and $(\mathbf{c}, f(\mathbf{c}))$, with vector $\mathbf{q}=\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$ depending on $f, \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, as the classical difference scalar quotient $q=q_{(g, a, b)}=\frac{g(b)-g(a)}{b-a}$ in (3) depends on $g, a$, and $b$.

In this work we show ${ }^{5}$ that vector $\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$ can still be expressed as a difference quotient, provided the corresponding vector product is the Clifford's geometric vector product (geometric product, for short). In order to show this, we will explain, in the next Section, what the geometric product is, at least between vectors in the two-dimensional Euclidean space $\mathbb{E}_{2}$. Here, we anticipate a consequenc ${ }^{6}$ of Theorem 1, that allows to express the vector quotient $\mathbf{q}$ as a fully symmetric linear combination of the outward normals of the oriented triangle $T$ determined by points $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ :

$$
\begin{equation*}
\mathbf{q}=\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}=\frac{f(\mathbf{a})+f(\mathbf{b})}{2 \tau} \partial_{\mathbf{c}}^{\perp}+\frac{f(\mathbf{a})+f(\mathbf{c})}{2 \tau} \partial_{\mathbf{b}}^{\perp}+\frac{f(\mathbf{b})+f(\mathbf{c})}{2 \tau} \partial_{\mathbf{a}}^{\perp} \tag{5}
\end{equation*}
$$

where $\partial_{\mathbf{c}}=\mathbf{b}-\mathbf{a}, \partial_{\mathbf{b}}=\mathbf{a}-\mathbf{c}$, and $\partial_{\mathbf{a}}=\mathbf{c}-\mathbf{b}$, and $\tau$ is the area of the triangle $T$, as illustrated by the figure below

[^2]

Remark. The vector quotient $\mathbf{q}$ is fully symmetric in the following sense:

$$
\mathbf{q}_{\left(f, \mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)}\right)}=\mathbf{q}_{\left(f, \mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right)} \text { for each } \sigma \in \mathcal{S}_{3},
$$

where $\mathcal{S}_{3}$ is the symmetric group of all permutations of the set $\{1,2,3\}$.
Remark. The Schwarz tangent paradox tells us that, despite the limit (1) exists when $g \in C^{1}(I)$,

$$
\lim _{\substack{\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{x}_{\mathbf{0}} \\ \mathbf{a}, \mathbf{b}, \mathbf{c} \text { not collinear }}}^{\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}} \text { doesn't exist, }
$$

even when the non-linear two-variable function $f$ is $C^{\infty}\left(\mathbb{E}_{2}\right)$, as in the foregoing counterexample.

## 2 The geometric product of two vectors in $\mathbb{E}_{2}$.

There are many ways to define the geometric product in an arbitrary quadratic space ${ }^{7}$. Here, for simplicity, we limit ourselves to present the geometric product between vectors of a two-dimensional Euclidean space $\mathbb{E}_{2}$, having $\mathbf{u} \cdot \mathbf{v}$ as positive definite symmetric bilinear form, and $|\mathbf{v}|=\sqrt{\mathbf{v} \cdot \mathbf{v}}$ as norm (where $\mathbf{u}, \mathbf{v}$ are vectors of $\mathbb{E}_{2}$ ).

### 2.1 Axioms.

we denote by $\mathbb{G}_{2}$ the associative algebra of polynomials of vectors of $\mathbb{E}_{2}$ (polyvectors, for short) satisfying the following rules:
(A1) scalars are considered as poly-vectors of degree 0 , and $\mathbf{v}^{0}=1$; non-zero vectors of $\mathbb{E}_{2}$ are considered as poly-vectors of degree 1 ;
(A4) addition in $\mathbb{R}$, addition in $\mathbb{E}_{2}$, and addition in $\mathbb{G}_{2}$ they all coincide ${ }^{8}$; multiplication in $\mathbb{R}$, multiplication of a scalar and a vector in $\mathbb{E}_{2}$, and multiplication of poly-vectors in $\mathbb{G}_{2}$ they all coincid ${ }^{9}$;

[^3](A3) scalars commute with vectors, that is
$$
\alpha \mathbf{v}=\mathbf{v} \alpha
$$
for each scalar $\alpha \in \mathbb{R}$, and vector $\mathbf{v} \in \mathbb{E}_{2}$;
(A5) scalars and non-zero vectors of $\mathbb{E}_{2}$ are linearly independent 10 in $\mathbb{G}_{2}$.
(A6) the Euclidean quadratic form $|\mathbf{v}|^{2}=\mathbf{v} \cdot \mathbf{v}$ in $\mathbb{E}_{2}$ is the geometric product of the vector $\mathbf{v} \in \mathbb{E}_{2}$ with itself, that is
$$
|\mathbf{v}|^{2}=\mathbf{v} \mathbf{v}=\mathbf{v}^{2}
$$

Elements of $\mathbb{G}_{2}$ are called multivectors.

### 2.2 Basic properties.

A first consequence of the foregoing axioms concerns invertibility of non-zero vectors.

Proposition. Every non-zero vector $\mathbf{v} \in \mathbb{E}_{2}$ is invertible with respect to the geometric product, and $\mathbf{v}^{-1}=\frac{1}{|\mathbf{v}|^{2}} \mathbf{v}$ is still a vector.

Proof. If $|\mathbf{v}| \neq 0$, then $\mathbf{v} \mathbf{v}^{-1}=\mathbf{v} \frac{1}{|\mathbf{v}|^{2}} \mathbf{v}=\frac{1}{|\mathbf{v}|^{2}} \mathbf{v} \mathbf{v}=1$.
Remark. By rule A3, notations like $\frac{\mathbf{v}}{|\mathbf{v}|}$ or $\frac{\mathbf{v}}{|\mathbf{v}|^{2}}$ are unambiguous. So, we can write $\mathbf{v}^{-1}=\frac{\mathbf{v}}{|\mathbf{v}|^{2}}$.

Axioms allow to clarify the nature of a particular symmetric poly-vector.
Proposition. The poly-vector $\frac{1}{2}(\mathbf{u v}+\mathbf{v u})$ is a number, whatever are the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{E}_{2}$.

Proof. By axiom A6, we have that $(\mathbf{u}+\mathbf{v})^{2}=|\mathbf{u}+\mathbf{v}|^{2}$ is a scalar. Moreover,

$$
(\mathbf{u}+\mathbf{v})^{2}=(\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v})=\mathbf{u} \mathbf{u}+\mathbf{u} \mathbf{v}+\mathbf{v} \mathbf{u}+\mathbf{v} \mathbf{v}=|\mathbf{u}|^{2}+\mathbf{u} \mathbf{v}+\mathbf{v} \mathbf{u}+|\mathbf{v}|^{2} .
$$

As also $|\mathbf{u}|^{2}$ and $|\mathbf{v}|^{2}$ are scalars, then we have that

$$
\mathbf{u} \mathbf{v}+\mathbf{v} \mathbf{u}=(\mathbf{u}+\mathbf{v})^{2}-|\mathbf{u}|^{2}-|\mathbf{v}|^{2} \in \mathbb{R}
$$

[^4]Remark 1. You can verify that the form $\beta$, corresponding to the foregoing poly-vector $\beta(\mathbf{u}, \mathbf{v})=\frac{1}{2}(\mathbf{u v}+\mathbf{v u})$, is bilinear, symmetric and coincide with the Euclidean scalar product $\mathbf{u} \cdot \mathbf{v}$, that is $\frac{1}{2}(\mathbf{u} \mathbf{v}+\mathbf{v u})=\mathbf{u} \cdot \mathbf{v}$, for each couple of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{E}_{2}$. This allows to control the non-commutativity of the geometric product between vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{E}_{2}$ as follows

$$
\mathbf{v u}=2(\mathbf{u} \cdot \mathbf{v})-\mathbf{u v}
$$

As

$$
\mathbf{u v}=\frac{1}{2}(\mathbf{u} \mathbf{v}+\mathbf{v u})+\frac{1}{2}(\mathbf{u v}-\mathbf{v} \mathbf{u})=(\mathbf{u} \cdot \mathbf{v})+\frac{1}{2}(\mathbf{u} \mathbf{v}-\mathbf{v u})
$$

then geometric product will be completely explained once the nature of the poly-vector

$$
\mathbf{u} \wedge \mathbf{v}=\frac{1}{2}(\mathbf{u} \mathbf{v}-\mathbf{v u})
$$

is clarified.
We first notice that $\mathbf{u} \wedge \mathbf{u}=0, \mathbf{u} \wedge \mathbf{v}=-\mathbf{v} \wedge \mathbf{u}$, and $\mathbf{u} \wedge \mathbf{v}$ is bilinear, that is

$$
(\alpha \mathbf{u}) \wedge \mathbf{v}=\mathbf{u} \wedge(\alpha \mathbf{v})=\alpha(\mathbf{u} \wedge \mathbf{v}), \text { and } \mathbf{u} \wedge(\mathbf{v}+\mathbf{w})=(\mathbf{u} \wedge \mathbf{v})+(\mathbf{u} \wedge \mathbf{w})
$$

for every scalar $\alpha$, and vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $\mathbb{E}_{2}$.
In order to investigate the nature of the multivector $\mathbf{u} \wedge \mathbf{v}$, let us observe that, if $\left\{\mathbf{g}_{1}, \mathbf{g}_{2}\right\}$ is an orthogonal basis for $\mathbb{E}_{2}, \mathbf{u}=\mu_{1} \mathbf{g}_{1}+\mu_{2} \mathbf{g}_{2}$, and $\mathbf{v}=\nu_{1} \mathbf{g}_{1}+\nu_{2} \mathbf{g}_{2}$, then

$$
\mathbf{g}_{1} \cdot \mathbf{g}_{2}=0 \quad, \quad \mathbf{g}_{1} \wedge \mathbf{g}_{2}=\mathbf{g}_{1} \mathbf{g}_{2} \quad, \quad \text { and } \quad \mathbf{u} \wedge \mathbf{v}=\operatorname{det}\left(\begin{array}{cc}
\mu_{1} & \mu_{2}  \tag{6}\\
\nu_{1} & \nu_{2}
\end{array}\right) \mathbf{g}_{1} \mathbf{g}_{2}
$$

We now prove that the dimension of the associative vector algebra $\mathbb{G}_{2}$ is 4 .
Proposition 1. If $\left\{\mathbf{g}_{1}, \mathbf{g}_{2}\right\}$ is an orthogonal basis of $\mathbb{E}_{2}$, then $\left\{1, \mathbf{g}_{1}, \mathbf{g}_{2}, \mathbf{g}_{1} \mathbf{g}_{2}\right\}$ is a basis of $\mathbb{G}_{2}$.

Proof. Let us prove that $\left\{1, \mathbf{g}_{1}, \mathbf{g}_{2}, \mathbf{g}_{1} \mathbf{g}_{2}\right\}$ generates $\mathbb{G}_{2}$.
Every element in $\mathbb{G}_{2}$ is a polynomial of vectors, that is a linear combination of product of vectors. As every vector is a linear combination of vectors $\mathbf{g}_{1}$, and $\mathbf{g}_{2}$, by the distributivity property, every product of vector can be written as a linear combination of product of vectors $\mathbf{g}_{1}$, and $\mathbf{g}_{2}$. As vectors $\mathbf{g}_{1}$, and $\mathbf{g}_{2}$ anticommute ( $\mathbf{g}_{1} \mathbf{g}_{2}=-\mathbf{g}_{2} \mathbf{g}_{1}$ ), every product of vectors $\mathbf{g}_{1}$, and $\mathbf{g}_{2}$ can be reduced to the product $\epsilon\left(\mathbf{g}_{1}\right)^{h}\left(\mathbf{g}_{2}\right)^{k}$ for some integer exponents $h, k \in \mathbb{N}$ and $\epsilon \in\{1,-1\}$. As $\left(\mathbf{g}_{i}\right)^{l}$ is a scalar if the integer exponent $l$ is even, and it is a multiple of vector $\mathbf{g}_{i}$ if $l$ is odd, then every poly-vector can be rewritten as a linear combination of elements in $\left\{1, \mathbf{g}_{1}, \mathbf{g}_{2}, \mathbf{g}_{1} \mathbf{g}_{2}\right\}$. This proves that $\left\{1, \mathbf{g}_{1}, \mathbf{g}_{2}, \mathbf{g}_{1} \mathbf{g}_{2}\right\}$ generates $\mathbb{G}_{2}$. It remains to show that $1, \mathbf{g}_{1}, \mathbf{g}_{2}$, and $\mathbf{g}_{1} \mathbf{g}_{2}$ are linearly independent. This means that we have to show that, if

$$
\begin{equation*}
\alpha_{0}+\alpha_{1} \mathbf{g}_{1}+\alpha_{2} \mathbf{g}_{2}+\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}=0 \tag{7}
\end{equation*}
$$

then necessarily $\alpha_{0}=\alpha_{1}=\alpha_{2}=\alpha_{3}=0$. First of all, let us observe that $\mathbf{g}_{1} \mathbf{g}_{2} \neq 0$. By contradiction, if $\mathbf{g}_{1} \mathbf{g}_{2}=0$, then by multiplying it from left by $\left(\mathbf{g}_{1}\right)^{-1}$ (recall that both vectors $\mathbf{g}_{1}$, and $\mathbf{g}_{2}$ are invertible), we would obtain $\mathbf{g}_{2}=0$, a contradiction with axiom A5. Let us now multiply the expression (7) from left by $\mathbf{g}_{1}$, and from right by $\left(\mathbf{g}_{1}\right)^{-1}$. Then, we can write the following equivalent relations

$$
\begin{gathered}
\mathbf{g}_{1}\left(\alpha_{0}+\alpha_{1} \mathbf{g}_{1}+\alpha_{2} \mathbf{g}_{2}+\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}\right)\left(\mathbf{g}_{1}\right)^{-1}=0 \\
\mathbf{g}_{1} \alpha_{0}\left(\mathbf{g}_{1}\right)^{-1}+\mathbf{g}_{1} \alpha_{1} \mathbf{g}_{1}\left(\mathbf{g}_{1}\right)^{-1}+\mathbf{g}_{1} \alpha_{2} \mathbf{g}_{2}\left(\mathbf{g}_{1}\right)^{-1}+\mathbf{g}_{1} \alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}\left(\mathbf{g}_{1}\right)^{-1}=0 \\
\alpha_{0}+\alpha_{1} \mathbf{g}_{1}+\alpha_{2} \mathbf{g}_{1} \mathbf{g}_{2}\left(\mathbf{g}_{1}\right)^{-1}+\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{1} \mathbf{g}_{2}\left(\mathbf{g}_{1}\right)^{-1}=0 \\
\alpha_{0}+\alpha_{1} \mathbf{g}_{1}-\alpha_{2} \mathbf{g}_{2} \mathbf{g}_{1}\left(\mathbf{g}_{1}\right)^{-1}-\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2} \mathbf{g}_{1}\left(\mathbf{g}_{1}\right)^{-1}=0 \\
\alpha_{0}+\alpha_{1} \mathbf{g}_{1}-\alpha_{2} \mathbf{g}_{2}-\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}=0
\end{gathered}
$$

By summing the last relation with (7), and then dividing the result by two, we obtain that

$$
\alpha_{0}+\alpha_{1} \mathbf{g}_{1}=0
$$

Rule A5 implies that $\alpha_{0}=\alpha_{1}=0$. So, the initial expression (7) reduces to

$$
\begin{equation*}
\alpha_{2} \mathbf{g}_{2}+\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}=0 \tag{8}
\end{equation*}
$$

Let us multiply the foregoing expression from left by $\mathbf{g}_{2}$, and from right by $\left(\mathbf{g}_{2}\right)^{-1}$. Then, we can write the following equivalent relations

$$
\begin{gathered}
\mathbf{g}_{2} \alpha_{2} \mathbf{g}_{2}\left(\mathbf{g}_{2}\right)^{-1}+\mathbf{g}_{2} \alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}\left(\mathbf{g}_{2}\right)^{-1}=0 \\
\alpha_{2} \mathbf{g}_{2}+\alpha_{3} \mathbf{g}_{2} \mathbf{g}_{1}=0 \\
\alpha_{2} \mathbf{g}_{2}-\alpha_{3} \mathbf{g}_{1} \mathbf{g}_{2}=0
\end{gathered}
$$

As before, by summing the last relation with (8), and then dividing the result by two, we obtain that $\alpha_{2}=0$. So, we end to the expression $\alpha_{3} \mathbf{g g}_{2}=0$. By multiplying it from left by $\left(\mathbf{g}_{2}\right)^{-1}\left(\mathbf{g}_{1}\right)^{-1}$, we obtain $\alpha_{3}=0$.

The foregoing result and (6) imply that $\mathbf{u} \wedge \mathbf{v}=0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. We can also notice that the geometric product of two orthonormal vectors does not depend on those particular factors, but only on their order.

Proposition. If the basis $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ for $\mathbb{E}_{2}$ is not only orthogonal, but also orthonormal (that is, $\left(\mathbf{e}_{1}\right)^{2}=\left(\mathbf{e}_{2}\right)^{2}=1$ ), then the geometric product $\mathbf{e}_{1} \mathbf{e}_{2} \in \mathbb{G}_{2}$ does not depend on the particular orthonormal basis $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$, but only on its orientation (that is, on the order the two vectors of the basis are given).

Proof. If $\left\{\mathbf{e}_{1}^{\prime}, \mathbf{e}_{2}^{\prime}\right\}$ is any other orthonormal basis for $\mathbb{E}_{2}, \mathbf{e}_{1}^{\prime}=\epsilon_{1,1} \mathbf{e}_{1}+\epsilon_{1,2} \mathbf{e}_{2}$ and $\mathbf{e}_{2}^{\prime}=\epsilon_{2,1} \mathbf{e}_{1}+\epsilon_{2,2} \mathbf{e}_{2}$, then

$$
\mathbf{e}_{1}^{\prime} \mathbf{e}_{2}^{\prime}=\operatorname{det}\left(\begin{array}{ll}
\epsilon_{1,1} & \epsilon_{1,2} \\
\epsilon_{2,1} & \epsilon_{2,2}
\end{array}\right) \mathbf{e}_{1} \mathbf{e}_{2}= \pm \mathbf{e}_{1} \mathbf{e}_{2}
$$

as $\left(\begin{array}{ll}\epsilon_{1,1} & \epsilon_{1,2} \\ \epsilon_{2,1} & \epsilon_{2,2}\end{array}\right)$ is an orthogonal matrix.
That is why we can denote the multivector $\mathbf{e}_{1} \mathbf{e}_{2}=I_{2}$, and call it an orientation of $\mathbb{E}_{2}$. So, by Proposition the basic multivectors in $\mathbb{G}_{2}$ are scalars, vectors, and multiples of orientations. You can also notice that

$$
\begin{aligned}
\left(I_{2}\right)^{2}=I_{2} I_{2}=\mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{1} \mathbf{e}_{2} & =\mathbf{e}_{1}\left(\mathbf{e}_{2} \mathbf{e}_{1}\right) \mathbf{e}_{2}=\mathbf{e}_{1}\left(-\mathbf{e}_{1} \mathbf{e}_{2}\right) \mathbf{e}_{2}=-\mathbf{e}_{1} \mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{2}= \\
& -\left(\mathbf{e}_{1} \mathbf{e}_{1}\right)\left(\mathbf{e}_{2} \mathbf{e}_{2}\right)=-1
\end{aligned}
$$

So, we have that $I_{2}$ is invertible in $\mathbb{G}_{2}$ and $\left(I_{2}\right)^{-1}=-I_{2}$. Doesn't $I_{2}$ strongly resemble the imaginary unit ?

Finally, we can write the geometric product between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{E}_{2}$ in terms of their coordinates with respect to any orthonormal basis $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ of $\mathbb{E}_{2}$,

$$
\begin{aligned}
\mathbf{u v} & =(\mathbf{u} \cdot \mathbf{v})+\mathbf{u} \wedge \mathbf{v}=\left(\mu_{1} \nu_{1}+\mu_{2} \nu_{2}\right)+\operatorname{det}\left(\begin{array}{ll}
\mu_{1} & \mu_{2} \\
\nu_{1} & \nu_{2}
\end{array}\right) I_{2} \\
& =|\mathbf{u}||\mathbf{v}|\left(\cos \theta+I_{2} \sin \theta\right)
\end{aligned}
$$

where $\mathbf{u}=\mu_{1} \mathbf{e}_{1}+\mu_{2} \mathbf{e}-2$, and $\mathbf{v}=\nu_{1} \mathbf{e}_{1}+\nu_{2} \mathbf{e}_{2}$, angle $\theta$ is oriented in $\mathbb{E}_{2}$ so that vector $\mathbf{u}$ can rotate towards vector $\mathbf{v}$ spanning angle $\theta \in(0, \pi)$, provided $\mathbf{u}$ and $\mathbf{v}$ are linearly independent (otherwise, $\mathbf{u v}=\mathbf{u} \cdot \mathbf{v}$ ).

### 2.3 Comparing vectors in $\mathbb{E}_{2} \subset \mathbb{G}_{2}$ with complex numbers.

You have probably remarked some similarities with complex numbers ( $I_{2}$ algebraically behaves in $\mathbb{G}_{2}$ as the imaginary unit $i$ does in $\mathbb{C}$ ). However, you should notice that while $\mathbb{G}_{2}$ has real-dimension four, $\mathbb{C}$ has real-dimension two. That is, scalars, vectors and orientations are distinguished in $\mathbb{G}_{2}$, while in $\mathbb{C}$ they all collapse into the notion of "complex number", viewed as a two-dimensional real vector. In other words, the geometric product of two vectors in $\mathbb{E}_{2}$ is not itself a vector: it is the sum of a scalar and a multiple of the orientation $I_{2}$. Instead, the product of two complex numbers (viewed as vectors in $\mathbb{R}^{2}$ ) is still a complex number (i.e., a vector in $\mathbb{R}^{2}$ ). Moreover, $\mathbb{C}$ is a division algebra (every non-zero element is invertible), while $\mathbb{G}_{2}$ is not: $1+\mathbf{e}_{1}$, and $1-\mathbf{e}_{1}$ are not invertible in $\mathbb{G}_{2}$, as $\left(1+\mathbf{e}_{1}\right)\left(1-\mathbf{e}_{1}\right)=0$. The product in $\mathbb{C}$ is commutative, while the geometric product in $\mathbb{G}_{2}$ is not. The geometric product in $\mathbb{G}_{2}$ is invariant by rotations in $\mathbb{E}_{2}$, while the product in $\mathbb{C}$ is not invariant by rotations in $\mathbb{R}^{2}$.

### 2.4 The determinant of a $2 \times 2$ real matrix viewed in $\mathbb{G}_{2}$.

Let us notice that, for every vector $\mathbf{v} \in \mathbb{E}_{2}$, then

$$
\begin{aligned}
\mathbf{v} I_{2} & =\left(\nu_{1} \mathbf{e}_{1}+\nu_{2} \mathbf{e}_{2}\right) \mathbf{e}_{1} \mathbf{e}_{2}=\nu_{1} \mathbf{e}_{1} \mathbf{e}_{1} \mathbf{e}_{2}+\nu_{2} \mathbf{e}_{2} \mathbf{e}_{1} \mathbf{e}_{2}=-\nu_{1} \mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{1}-\nu_{2} \mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{2} \\
& =-\mathbf{e}_{1} \mathbf{e}_{2}\left(\nu_{1} \mathbf{e}_{1}+\nu_{2} \mathbf{e}_{2}\right)=-I_{2} \mathbf{v}=\nu_{1} \mathbf{e}_{2}-\nu_{2} \mathbf{e}_{1} \in \mathbb{E}_{2}
\end{aligned}
$$

which corresponds to the vector obtained by rotating vector $\mathbf{v}$ of a right angle according to the orientation $I_{2}=\mathbf{e}_{1} \mathbf{e}_{2}$, as illustrated below,


Thanks to the foregoing results we can now proceed to show the two key facts that link the geometric product to the basic notion of secant plane.

Proposition. The determinant of a $2 x 2$ real matrix is a Clifford ratio in $\mathbb{G}_{2}$.
Proof. Let us consider the rows of a $2 \times 2$ real matrix

$$
\left(\begin{array}{ll}
\mu_{1} & \mu_{2} \\
\nu_{1} & \nu_{2}
\end{array}\right)
$$

as the components of two vectors $\mathbf{u}=\mu_{1} \mathbf{e}_{1}+\mu_{2} \mathbf{e}_{2}$, and $\mathbf{v}=\nu_{1} \mathbf{e}_{1}+\nu_{2} \mathbf{e}_{2}$ in $\mathbb{E}_{2}$ with respect to some orthonormal basis $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$.

As $\mathbf{u} \wedge \mathbf{v}=\operatorname{det}\left(\begin{array}{cc}\mu_{1} & \mu_{2} \\ \nu_{1} & \nu_{2}\end{array}\right) \mathbf{e}_{1} \mathbf{e}_{2}=\operatorname{det}\left(\begin{array}{cc}\mu_{1} & \mu_{2} \\ \nu_{1} & \nu_{2}\end{array}\right) I_{2}$, then we can write

$$
\operatorname{det}\left(\begin{array}{cc}
\mu_{1} & \mu_{2} \\
\nu_{1} & \nu_{2}
\end{array}\right)=(\mathbf{u} \wedge \mathbf{v})\left(I_{2}\right)^{-1}, \text { which is a ratio in } \mathbb{G}_{2} .
$$

Remark. You can verify that $(\mathbf{u} \wedge \mathbf{v})\left(I_{2}\right)^{-1}=\left(I_{2}\right)^{-1}(\mathbf{u} \wedge \mathbf{v})$. So the expression $\operatorname{det}\left(\begin{array}{cc}\mu_{1} & \mu_{2} \\ \nu_{1} & \nu_{2}\end{array}\right)=\frac{\mathbf{u} \wedge \mathbf{v}}{I_{2}}$ for the determinant of a $2 \times 2$ matrix is unambiguous.

Proposition 2. The determinant a 2x2 real matrix is a scalar product.
Proof. By using the same assumptions of the foregoing proof, we have that

$$
\begin{aligned}
\operatorname{det}\left(\begin{array}{cc}
\mu_{1} & \mu_{2} \\
\nu_{1} & \nu_{2}
\end{array}\right) & =(\mathbf{u} \wedge \mathbf{v})\left(I_{2}\right)^{-1}=-(\mathbf{u} \wedge \mathbf{v}) I_{2}=(\mathbf{v} \wedge \mathbf{u}) I_{2}=\frac{1}{2}(\mathbf{v u}-\mathbf{u v}) I_{2} \\
& =\frac{1}{2}\left(\mathbf{v} \mathbf{u} I_{2}-\mathbf{u v} I_{2}\right)=\frac{1}{2}\left[\mathbf{v}\left(\mathbf{u} I_{2}\right)+\left(\mathbf{u} I_{2}\right) \mathbf{u}\right]=\left(\mathbf{u} I_{2}\right) \cdot \mathbf{v}
\end{aligned}
$$

## 3 The difference vector quotient of a secant plane.

Let us reformulate the relations defining the plane secant the graph of a function. In the one-variable case the equation of the line secant the graph of $g: I \rightarrow \mathbb{R}$ at
points $(a, g(a)),(b, g(b)) \in I \times \mathbb{R}$ can be written, in the Cartesian $(x, z)$-plane, in two equivalent ways: (3) or

$$
\operatorname{det}\left(\begin{array}{cc}
x-a & z-g(a) \\
b-a & g(b)-g(a)
\end{array}\right)=0
$$

In the $(x, y, z)$-coordinate system, the equation of the plane secant the graph of $f$ at points $\left(\alpha_{1}, \alpha_{2}, f(\mathbf{a})\right),\left(\beta_{1}, \beta_{2}, f(\mathbf{b})\right),\left(\gamma_{1}, \gamma_{2}, f(\mathbf{c})\right) \in \mathbb{R}^{3}$ can be written as

$$
\operatorname{det}\left(\begin{array}{ccc}
x-\alpha_{1} & y-\alpha_{2} & z-f(\mathbf{a})  \tag{9}\\
\beta_{1}-\alpha_{1} & \beta_{2}-\alpha_{2} & f(\mathbf{b})-f(\mathbf{a}) \\
\gamma_{1}-\alpha_{1} & \gamma_{2}-\alpha_{2} & f(\mathbf{c})-f(\mathbf{a})
\end{array}\right)=0
$$

where $\mathbf{a}=\alpha_{1} \mathbf{e}_{1}+\alpha_{2} \mathbf{e}_{2}, \mathbf{b}=\beta_{1} \mathbf{e}_{1}+\beta_{2} \mathbf{e}_{2}, \mathbf{c}=\gamma_{1} \mathbf{e}_{1}+\gamma_{2} \mathbf{e}_{2}$ are vectors (that we also call points) in the two-dimensional domain $\Omega \subseteq \mathbb{E}_{2}$ of $f$, and $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ is an orthonormal basis of $\mathbb{E}_{2}$. In the following, we show how to rewrite (9) in the $\mathbb{E}_{2}$-coordinate-free setting (4). The following equivalences start from a Laplace expansion of the determinant (9).

$$
\begin{gathered}
(z-f(\mathbf{a})) \operatorname{det}\left(\begin{array}{cc}
\beta_{1}-\alpha_{1} & \beta_{2}-\alpha_{2} \\
\gamma_{1}-\alpha_{1} & \gamma_{2}-\alpha_{2}
\end{array}\right)-(f(\mathbf{b})-f(\mathbf{a})) \operatorname{det}\left(\begin{array}{cc}
x-\alpha_{1} & y-\alpha_{2} \\
\gamma_{1}-\alpha_{1} & \gamma_{2}-\alpha_{2}
\end{array}\right)+ \\
+(f(\mathbf{c})-f(\mathbf{a})) \operatorname{det}\left(\begin{array}{cc}
x-\alpha_{1} & y-\alpha_{2} \\
\beta_{1}-\alpha_{1} & \beta_{2}-\alpha_{2}
\end{array}\right)=0 \\
(f(\mathbf{b})-f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]\left(I_{2}\right)^{-1}-(f(\mathbf{c})-f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{b}-\mathbf{a})]\left(I_{2}\right)^{-1} \\
(z-f(\mathbf{a}))[(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]= \\
(f(\mathbf{b})-f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]-(f(\mathbf{c})-f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{b}-\mathbf{a})] \\
z-f(\mathbf{a})=(f(\mathbf{b})-f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})][(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]^{-1}+-(f(\mathbf{c})- \\
f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{b}-\mathbf{a})][(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]^{-1} \\
z=f(\mathbf{a})+(f(\mathbf{b})-f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})][(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]^{-1}+-(f(\mathbf{c})- \\
f(\mathbf{a}))[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{b}-\mathbf{a})][(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]^{-1}
\end{gathered}
$$

As $(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})=\mathbf{a} \wedge \mathbf{b}+\mathbf{b} \wedge \mathbf{c}+\mathbf{c} \wedge \mathbf{a}$

$$
=I_{2} \underbrace{\left[\operatorname{det}\left(\begin{array}{cc}
\alpha_{1} & \alpha_{2} \\
\beta_{1} & \beta_{2}
\end{array}\right)+\operatorname{det}\left(\begin{array}{ll}
\beta_{1} & \beta_{2} \\
\gamma_{1} & \gamma_{2}
\end{array}\right)+\operatorname{det}\left(\begin{array}{cc}
\gamma_{1} & \gamma_{2} \\
\alpha_{1} & \alpha_{2}
\end{array}\right)\right]}_{2 \tau}
$$

where $\tau$ is simply the oriented 11 area of the triangle having as vertices the points $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ ), the foregoing equivalences, describing the secant plane, can continue as follows
${ }^{11}$ The sign of $\tau$ is positive if and only if the geometric ratio between $(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})$ and
$I_{2}$ is positive. See also [1].

$$
\begin{aligned}
& z=f(\mathbf{a})+\frac{f(\mathbf{b})-f(\mathbf{a})}{2 \tau}[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})] I_{2}^{-1}-\frac{f(\mathbf{c})-f(\mathbf{a})}{2 \tau}[(\mathbf{x}-\mathbf{a}) \wedge(\mathbf{b}-\mathbf{a})] I_{2}^{-1} \\
& z=f(\mathbf{a})-\frac{f(\mathbf{b})-f(\mathbf{a})}{2 \tau}[(\mathbf{c}-\mathbf{a}) \wedge(\mathbf{x}-\mathbf{a})] I_{2}^{-1}+\frac{f(\mathbf{c})-f(\mathbf{a})}{2 \tau}[(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{x}-\mathbf{a})] I_{2}^{-1}
\end{aligned}
$$

By proposition 2, we can write

$$
\begin{gathered}
z=f(\mathbf{a})-\frac{f(\mathbf{b})-f(\mathbf{a})}{2 \tau}\left\{\left[(\mathbf{c}-\mathbf{a}) I_{2}\right] \cdot(\mathbf{x}-\mathbf{a})\right\}+\frac{f(\mathbf{c})-f(\mathbf{a})}{2 \tau}\left\{\left[(\mathbf{b}-\mathbf{a}) I_{2}\right] \cdot(\mathbf{x}-\mathbf{a})\right\} \\
z=f(\mathbf{a})-\left\{\frac{f(\mathbf{b})-f(\mathbf{a})}{2 \tau}\left[(\mathbf{c}-\mathbf{a}) I_{2}\right]-\frac{f(\mathbf{c})-f(\mathbf{a})}{2 \tau}\left[(\mathbf{b}-\mathbf{a}) I_{2}\right]\right\} \cdot(\mathbf{x}-\mathbf{a}) \\
z=f(\mathbf{a})-\frac{1}{2 \tau}\left\{[(f(\mathbf{b})-f(\mathbf{a}))(\mathbf{c}-\mathbf{a})-(f(\mathbf{c})-f(\mathbf{a}))(\mathbf{b}-\mathbf{a})] I_{2}\right\} \cdot(\mathbf{x}-\mathbf{a})
\end{gathered}
$$

This allows to explicitly write vector $\mathbf{q}=\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$ of expression (44)

$$
\begin{aligned}
\mathbf{q} & =-\frac{1}{2 \tau}\left\{[(f(\mathbf{b})-f(\mathbf{a}))(\mathbf{c}-\mathbf{a})-(f(\mathbf{c})-f(\mathbf{a}))(\mathbf{b}-\mathbf{a})] I_{2}\right. \\
& =[(f(\mathbf{b})-f(\mathbf{a}))(\mathbf{c}-\mathbf{a})-(f(\mathbf{c})-f(\mathbf{a}))(\mathbf{b}-\mathbf{a})][(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})]^{-1},
\end{aligned}
$$

and proves the following main result of this article.
Theorem 1. The plane secant the graph of a two-variable function $f: \Omega \subseteq$ $\mathbb{E}_{2} \rightarrow \mathbb{R}$ at points $(\mathbf{a}, f(\mathbf{a})),(\mathbf{b}, f(\mathbf{b}))$, and $(\mathbf{c}, f(\mathbf{c}))$ (where $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are three non-collinear points in the domain $\Omega$ of $f$ ), can be represented in the $(\mathbf{v}, z)$-space $\mathbb{E}_{2} \times \mathbb{R}$ by the relation

$$
z=f(\mathbf{a})+\mathbf{q} \cdot(\mathbf{v}-\mathbf{a})
$$

where vector $\mathbf{q}=\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$ is the geometric quotient in $\mathbb{G}_{2}$ between the vector

$$
(f(\mathbf{b})-f(\mathbf{a}))(\mathbf{c}-\mathbf{a})-(f(\mathbf{c})-f(\mathbf{a}))(\mathbf{b}-\mathbf{a}),
$$

and the orientation

$$
(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})
$$

## 4 The vector quotient as a linear combination in $\mathbb{E}_{2}$.

Computations in $\mathbb{G}_{2}$ allow to explicitly express the foregoing difference vector quotient $\mathbf{q}=\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$ as a linear combination of normals to the triangle defined by $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$.

Let $\partial_{\mathbf{a}}=\mathbf{c}-\mathbf{b}, \partial_{\mathbf{b}}=\mathbf{a}-\mathbf{c}, \partial_{\mathbf{c}}=\mathbf{b}-\mathbf{a}$; so, $\partial_{\mathbf{a}}+\partial_{\mathbf{b}}+\partial_{\mathbf{c}}=0$. Moreover

$$
(\mathbf{b}-\mathbf{a}) \wedge(\mathbf{c}-\mathbf{a})=\partial_{\mathbf{a}} \wedge \partial_{\mathbf{b}}=\partial_{\mathbf{b}} \wedge \partial_{\mathbf{c}}=\partial_{\mathbf{c}} \wedge \partial_{\mathbf{a}}=2 \tau I_{2}
$$

where $\tau$ is the signed area of the oriented triangle determined by the ordered points $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, whose sign depends on the orientation $I_{2}$. Then,

$$
\begin{aligned}
& (f(\mathbf{b})-f(\mathbf{a}))(\mathbf{c}-\mathbf{a})-(f(\mathbf{c})-f(\mathbf{a}))(\mathbf{b}-\mathbf{a})=(f(\mathbf{a})-f(\mathbf{b})) \partial_{\mathbf{b}}-(f(\mathbf{c})-f(\mathbf{a})) \partial_{\mathbf{c}} \\
= & f(\mathbf{a})\left[\partial_{\mathbf{b}}+\partial_{\mathbf{c}}\right]-f(\mathbf{b}) \partial_{\mathbf{b}}-f(\mathbf{c}) \partial_{\mathbf{c}}=-\left[f(\mathbf{a}) \partial_{\mathbf{a}}+f(\mathbf{b}) \partial_{\mathbf{b}}+f(\mathbf{c}) \partial_{\mathbf{c}}\right] \\
= & f(\mathbf{a})\left(\partial_{\mathbf{b}}+\partial_{\mathbf{c}}\right)+f(\mathbf{b})\left(\partial_{\mathbf{a}}+\partial_{\mathbf{c}}\right)+f(\mathbf{c})\left(\partial_{\mathbf{a}}+\partial_{\mathbf{b}}\right) \\
= & {[f(\mathbf{a})+f(\mathbf{b})] \partial_{\mathbf{c}}+[f(\mathbf{a})+f(\mathbf{c})] \partial_{\mathbf{b}}+[f(\mathbf{b})+f(\mathbf{c b})] \partial_{\mathbf{a}} }
\end{aligned}
$$

So, we can write

$$
\begin{aligned}
\mathbf{q} & =\left\{[f(\mathbf{a})+f(\mathbf{b})] \partial_{\mathbf{c}}+[f(\mathbf{a})+f(\mathbf{c})] \partial_{\mathbf{b}}+[f(\mathbf{b})+f(\mathbf{c})] \partial_{\mathbf{a}}\right\}\left(2 \tau I_{2}\right)^{-1} \\
& =\frac{f(\mathbf{a})+f(\mathbf{b})}{2 \tau} \partial_{\mathbf{c}}\left(I_{2}\right)^{-1}+\frac{f(\mathbf{a})+f(\mathbf{c})}{2 \tau} \partial_{\mathbf{b}}\left(I_{2}\right)^{-1}+\frac{f(\mathbf{b})+f(\mathbf{c})}{2 \tau} \partial_{\mathbf{a}}\left(I_{2}\right)^{-1} \\
& =\frac{f(\mathbf{a})+f(\mathbf{b})}{2 \tau} \underbrace{I_{2} \partial_{\mathbf{c}}}_{\partial_{\mathbf{c}}^{\perp}}+\frac{f(\mathbf{a})+f(\mathbf{c})}{2 \tau} \underbrace{I_{2} \partial_{\mathbf{b}}}_{\partial_{\mathbf{b}}^{\perp}}+\frac{f(\mathbf{b})+f(\mathbf{c})}{2 \tau} \underbrace{I_{2} \partial_{\mathbf{a}}}_{\partial_{\mathbf{a}}^{\perp}} .
\end{aligned}
$$

This proves our second theorem, which generalizes a lemma proved ${ }^{12}$ in 4.
Theorem 2. The plane secant the graph of a two-variable function $f: \Omega \subseteq$ $\mathbb{E}_{2} \rightarrow \mathbb{R}$ at points $(\mathbf{a}, f(\mathbf{a})),(\mathbf{b}, f(\mathbf{b}))$, and $(\mathbf{c}, f(\mathbf{c}))($ where $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are three non-collinear points in the domain $\Omega$ of $f$ ), can be represented in the $(\mathbf{v}, z)$-space $\mathbb{E}_{2} \times \mathbb{R}$ by the relation $z=f(\mathbf{a})+\mathbf{q} \cdot(\mathbf{v}-\mathbf{a})$, where vector $\mathbf{q}=\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$ is the linear combination (5).

Remark. If $f$ is the affine function $f(\mathbf{x})=(\mathbf{v} \cdot \mathbf{x})+\phi$, for some $\mathbf{v} \in \mathbb{E}_{2}$ and $\phi \in \mathbb{R}$, then you can verify that $\mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}=\mathbf{v}$.

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    ${ }^{1}$ See 13], 9], and 15].
    ${ }^{2}$ See 2 at pages 142-144.
    ${ }^{3}$ See [10, and 6.

[^1]:    ${ }^{4}$ Vectors, also called points, will be indicated by bold Latin letters; real numbers, also called scalars, will be denoted by lower-case Latin or Greek letters.

[^2]:    ${ }^{5}$ See Theorem 1 at the end of this article.
    ${ }^{6}$ See Theorem 2

[^3]:    ${ }^{7}$ See (12], [11], [3], [7], [5], 8], or [14], for instance.
    ${ }^{8}$ This implies that the zero scalar coincides with the null vector in $\mathbb{G}_{2}$. That is, $0=\mathbf{0}$.
    ${ }^{9}$ As a consequence of this axiom, it is natural to denote the geometric product by juxtaposition.

[^4]:    ${ }^{10}$ That is, $\alpha+\beta \mathbf{v}=0$ if and only if $\alpha=\beta=0$ (where $\alpha, \beta \in \mathbb{R}$, and $\mathbf{0} \neq \mathbf{v} \in \mathbb{E}_{2}$ ).

[^5]:    ${ }^{12}$ Only for linear functions.

