# The surface tangent paradox and the difference vector quotient of a secant plane

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#### Abstract

If a one-variable function is sufficiently smooth, then the limit position of secant lines its graph is a tangent line. By analogy, one would expect that the limit position of secant planes of a two-variable smooth function is a plane tangent to its graph. Amazingly, this is not necessarily true, even when the function is a simple polynomial. Despite this paradox, we show that some analogies with the one-variable case still hold in the multi-variable context, provided we use a particular vector product: the Clifford's geometric one.

### 1 Introduction.

The classical Schwarz surface area paradox<sup>1</sup> provides a counterexample concerning the definition of the area of a curved surface. It is a divergence phenomenon, that is seldom presented, even at the end of an Advanced Calculus course<sup>2</sup>. Here, we show that a local version of this paradox deals with the very definition of the tangent to the graph of a smooth function. For simplicity, we only consider functions of one and two real variables.

Let us first recall the link between lines secant and tangent the graph of a one-variable smooth function  $g: I \subseteq \mathbb{R} \to \mathbb{R}$ . If you consider the line passing through two distinct points  $(a, g(a)), (b, g(b)) \in I \times \mathbb{R}$  of the graph of g, then this secant line becomes the line tangent the graph of g at point  $(x_0, g(x_0))$ , as the distinct points a and b converge towards  $x_0$ , whatever is the way they do it, provided g is sufficiently smooth (a  $C^1$ -function, let's say). Analytically, this corresponds to the existence of the strong derivative<sup>3</sup> of g at point  $x_0$ , which coincides with the classical derivative  $g'(x_0)$ , provided the strong derivative

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<sup>&</sup>lt;sup>1</sup>See [13], [9], and [15].

 $<sup>^{2}</sup>$ See [2] at pages 142–144

 $<sup>^{3}</sup>$ See [10], and [6].

exists (as it happens for a  $C^1$ -function). More precisely,

$$\lim_{\substack{a,b\to x_0\\a\neq b}} \frac{g(b) - g(a)}{b - a} = g'(x_0) \tag{1}$$

By analogy, one would expect that, for a two-variable smooth function, the plane tangent to its graph at a fixed point would be the limit position of the secant plane passing through three non-collinear points, as these three non-collinear points converge to that fixed point. Paradoxically, this is not the case.

#### 1.1 Counterexample.

Let us consider as two-variable smooth function f the polynomial  $f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = x^2 + y^2$ , when  $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$ , and  $\{\mathbf{e}_1, \mathbf{e}_2\}$  is an orthonormal basis of the twodimensional vector<sup>4</sup> Euclidean space  $\mathbb{E}_2$ , with scalar product  $\mathbf{u} \cdot \mathbf{w}$ , with  $\mathbf{u}, \mathbf{w}$ vectors of  $\mathbb{E}_2$ . So,  $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ , and  $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1$ . If  $\mathbf{a} = \mathbf{0}$ ,  $\mathbf{b} = -\delta \mathbf{e}_1 + \eta \mathbf{e}_2$ , and  $\mathbf{c} = \delta \mathbf{e}_1 + \eta \mathbf{e}_2$ , then the Cartesian equation of the plane passing through points  $(\mathbf{a}, f(\mathbf{a}))$ ,  $(\mathbf{b}, f(\mathbf{b}))$ , and  $(\mathbf{c}, f(\mathbf{c}))$  can be represented, in the (x, y, z)coordinates, by the relation

$$\eta z = (\delta^2 + \eta^2) y \tag{2}$$

As the limit of  $\frac{\delta^2 + \eta^2}{\eta}$ , as  $\delta, \eta \to 0$ , does not exist, then the secant plane (2) does not necessarily converge to the plane z = 0, which is tangent to the graph of f at point **0**.

This counterexample illustrates the **tangent paradox**: a plane secant the graph of a smooth two-variable function f at points  $(\mathbf{a}, f(\mathbf{a}))$ ,  $(\mathbf{b}, f(\mathbf{b}))$ , and  $(\mathbf{c}, f(\mathbf{c}))$  can assume limit positions that depend on the way the three non-collinear points  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  converge to  $\mathbf{x}_0$ . For example, if you consider f,  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  as in the counterexample,

- if  $\eta = \delta \rightarrow 0$ , then the limit plane would be z = 0, the tangent plane;
- if  $\eta = \delta^2 \to 0$ , then the limit plane would be z = y, which is not the tangent plane;
- if  $\eta = \delta^3 \to 0$ , then the limit plane would be y = 0, which is even orthogonal to the tangent plane!

So, contrary to the one-variable case, we cannot simply say that the tangent plane to a smooth two-variable function is the limit position of its secant planes.

 $<sup>^{4}</sup>$ Vectors, also called *points*, will be indicated by **bold** Latin letters; real numbers, also called *scalars*, will be denoted by lower-case Latin or Greek letters.

#### 1.2 In search for analogies.

Despite such paradox, one can ask if any analogy with the one-variable case still holds for two-variable smooth functions.

In the (x, z)-plane, the Cartesian equation of the line passing through (a, g(a)), (b, g(b)) (with  $a \neq b$ ) is

$$z = g(a) + \frac{g(b) - g(a)}{b - a}(x - a)$$
(3)

The existence of the strong derivative (1), and the continuity of g at  $x_0$  imply that, as  $a, b \to x_0$ , relation (3) becomes relation

$$z = g(x_0) + g'(x_0)(x - x_0)$$
,

which represents the line tangent the graph of g at point  $(x_0, g(x_0))$ . The relation representing the plane tangent the graph of the two-variable function f:  $\Omega \subseteq \mathbb{E}_2 \to \mathbb{R}$  at point  $(\mathbf{v}_0, f(\mathbf{v}_0))$  can be written in the  $(\mathbf{v}, z)$ -space  $\mathbb{E}_2 \times \mathbb{R}$  as

$$z = f(\mathbf{v}_0) + \nabla f(\mathbf{v}_0) \cdot (\mathbf{v} - \mathbf{v}_0)$$

So, we wonder if, in analogy with (3), there exists a relation

$$z = f(\mathbf{a}) + \mathbf{q} \cdot (\mathbf{v} - \mathbf{a}) \tag{4}$$

representing the plane passing through  $(\mathbf{a}, f(\mathbf{a}))$ ,  $(\mathbf{b}, f(\mathbf{b}))$ , and  $(\mathbf{c}, f(\mathbf{c}))$ , with vector  $\mathbf{q} = \mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$  depending on f,  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ , as the classical difference scalar quotient  $q = q_{(g,a,b)} = \frac{g(b)-g(a)}{b-a}$  in (3) depends on g, a, and b. In this work we show<sup>5</sup> that vector  $\mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$  can still be expressed as a dif-

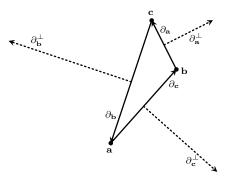
In this work we show<sup>5</sup> that vector  $\mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$  can still be expressed as a difference quotient, provided the corresponding vector product is the Clifford's geometric vector product (*geometric product*, for short). In order to show this, we will explain, in the next Section, what the geometric product is, at least between vectors in the two-dimensional Euclidean space  $\mathbb{E}_2$ . Here, we anticipate a consequence<sup>6</sup> of Theorem 1, that allows to express the vector quotient  $\mathbf{q}$  as a fully symmetric linear combination of the outward normals of the oriented triangle T determined by points  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ :

$$\mathbf{q} = \mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{f(\mathbf{a}) + f(\mathbf{b})}{2\tau} \partial_{\mathbf{c}}^{\perp} + \frac{f(\mathbf{a}) + f(\mathbf{c})}{2\tau} \partial_{\mathbf{b}}^{\perp} + \frac{f(\mathbf{b}) + f(\mathbf{c})}{2\tau} \partial_{\mathbf{a}}^{\perp} \quad (5)$$

where  $\partial_{\mathbf{c}} = \mathbf{b} - \mathbf{a}$ ,  $\partial_{\mathbf{b}} = \mathbf{a} - \mathbf{c}$ , and  $\partial_{\mathbf{a}} = \mathbf{c} - \mathbf{b}$ , and  $\tau$  is the area of the triangle T, as illustrated by the figure below

 $<sup>^5 \</sup>mathrm{See}$  Theorem 1 at the end of this article.

 $<sup>^6 \</sup>mathrm{See}$  Theorem 2.



**Remark.** The vector quotient **q** is fully symmetric in the following sense:

 $\mathbf{q}_{(f,\mathbf{v}_{\sigma(1)},\mathbf{v}_{\sigma(2)},\mathbf{v}_{\sigma(3)})} = \mathbf{q}_{(f,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} \text{ for each } \sigma \in \mathcal{S}_3,$ 

where  $S_3$  is the symmetric group of all permutations of the set  $\{1, 2, 3\}$ .

**Remark.** The Schwarz tangent paradox tells us that, despite the limit (1) exists when  $g \in C^1(I)$ ,

$$\lim_{\substack{\mathbf{a},\mathbf{b},\mathbf{c}\to\mathbf{x}_0\\\mathbf{b},\mathbf{c} \text{ not collinear}}} \mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} \quad \text{doesn't exist},$$

even when the non-linear two-variable function f is  $C^{\infty}(\mathbb{E}_2)$ , as in the foregoing counterexample.

# 2 The geometric product of two vectors in $\mathbb{E}_2$ .

There are many ways to define the geometric product in an arbitrary quadratic space<sup>7</sup>. Here, for simplicity, we limit ourselves to present the geometric product between vectors of a two-dimensional Euclidean space  $\mathbb{E}_2$ , having  $\mathbf{u} \cdot \mathbf{v}$  as positive definite symmetric bilinear form, and  $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$  as norm (where  $\mathbf{u}, \mathbf{v}$  are vectors of  $\mathbb{E}_2$ ).

#### 2.1 Axioms.

we denote by  $\mathbb{G}_2$  the associative algebra of polynomials of vectors of  $\mathbb{E}_2$  (polyvectors, for short) satisfying the following rules:

- (A1) scalars are considered as poly-vectors of degree 0, and  $\mathbf{v}^0 = 1$ ; non-zero vectors of  $\mathbb{E}_2$  are considered as poly-vectors of degree 1;
- (A4) addition in ℝ, addition in E<sub>2</sub>, and addition in G<sub>2</sub> they all coincide<sup>8</sup>; multiplication in ℝ, multiplication of a scalar and a vector in E<sub>2</sub>, and multiplication of poly-vectors in G<sub>2</sub> they all coincide<sup>9</sup>;

 $\mathbf{a}$ 

<sup>&</sup>lt;sup>7</sup>See [12], [11], [3], [7], [5], [8], or [14], for instance.

<sup>&</sup>lt;sup>8</sup>This implies that the zero scalar coincides with the null vector in  $\mathbb{G}_2$ . That is,  $0 = \mathbf{0}$ .

 $<sup>^{9}\</sup>mathrm{As}$  a consequence of this axiom, it is natural to denote the geometric product by juxta-position.

(A3) scalars commute with vectors, that is

 $\alpha \mathbf{v} = \mathbf{v} \alpha$ 

for each scalar  $\alpha \in \mathbb{R}$ , and vector  $\mathbf{v} \in \mathbb{E}_2$ ;

- (A5) scalars and non-zero vectors of  $\mathbb{E}_2$  are linearly independent<sup>10</sup> in  $\mathbb{G}_2$ .
- (A6) the Euclidean quadratic form  $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$  in  $\mathbb{E}_2$  is the geometric product of the vector  $\mathbf{v} \in \mathbb{E}_2$  with itself, that is

$$|\mathbf{v}|^2 = \mathbf{v}\mathbf{v} = \mathbf{v}^2 \ .$$

Elements of  $\mathbb{G}_2$  are called *multivectors*.

#### 2.2 Basic properties.

A first consequence of the foregoing axioms concerns invertibility of non-zero vectors.

**Proposition.** Every non-zero vector  $\mathbf{v} \in \mathbb{E}_2$  is invertible with respect to the geometric product, and  $\mathbf{v}^{-1} = \frac{1}{|\mathbf{v}|^2} \mathbf{v}$  is still a vector.

*Proof.* If 
$$|\mathbf{v}| \neq 0$$
, then  $\mathbf{v}\mathbf{v}^{-1} = \mathbf{v}\frac{1}{|\mathbf{v}|^2}\mathbf{v} = \frac{1}{|\mathbf{v}|^2}\mathbf{v}\mathbf{v} = 1$ .

**Remark.** By rule A3, notations like  $\frac{\mathbf{v}}{|\mathbf{v}|}$  or  $\frac{\mathbf{v}}{|\mathbf{v}|^2}$  are unambiguous. So, we can write  $\mathbf{v}^{-1} = \frac{\mathbf{v}}{|\mathbf{v}|^2}$ .

Axioms allow to clarify the nature of a particular symmetric poly-vector.

**Proposition.** The poly-vector  $\frac{1}{2}(\mathbf{uv}+\mathbf{vu})$  is a number, whatever are the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{E}_2$ .

*Proof.* By axiom A6, we have that  $(\mathbf{u} + \mathbf{v})^2 = |\mathbf{u} + \mathbf{v}|^2$  is a scalar. Moreover,

$$(\mathbf{u} + \mathbf{v})^2 = (\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v}) = \mathbf{u}\mathbf{u} + \mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} + \mathbf{v}\mathbf{v} = |\mathbf{u}|^2 + \mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} + |\mathbf{v}|^2 .$$

As also  $|\mathbf{u}|^2$  and  $|\mathbf{v}|^2$  are scalars, then we have that

$$\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} = (\mathbf{u} + \mathbf{v})^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2 \in \mathbb{R}$$

<sup>&</sup>lt;sup>10</sup>That is,  $\alpha + \beta \mathbf{v} = 0$  if and only if  $\alpha = \beta = 0$  (where  $\alpha, \beta \in \mathbb{R}$ , and  $\mathbf{0} \neq \mathbf{v} \in \mathbb{E}_2$ ).

**Remark 1.** You can verify that the form  $\beta$ , corresponding to the foregoing poly-vector  $\beta(\mathbf{u}, \mathbf{v}) = \frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})$ , is bilinear, symmetric and coincide with the Euclidean scalar product  $\mathbf{u} \cdot \mathbf{v}$ , that is  $\frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) = \mathbf{u} \cdot \mathbf{v}$ , for each couple of vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{E}_2$ . This allows to control the non-commutativity of the geometric product between vectors  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{E}_2$  as follows

$$\mathbf{v}\mathbf{u} = 2(\mathbf{u}\cdot\mathbf{v}) - \mathbf{u}\mathbf{v}$$

 $\mathbf{As}$ 

$$\mathbf{u}\mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) + \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}) = (\mathbf{u}\cdot\mathbf{v}) + \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u})$$

then geometric product will be completely explained once the nature of the poly-vector

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u})$$

is clarified.

We first notice that  $\mathbf{u} \wedge \mathbf{u} = 0$ ,  $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$ , and  $\mathbf{u} \wedge \mathbf{v}$  is bilinear, that is

$$(\alpha \mathbf{u}) \wedge \mathbf{v} = \mathbf{u} \wedge (\alpha \mathbf{v}) = \alpha(\mathbf{u} \wedge \mathbf{v}), \text{ and } \mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \wedge \mathbf{v}) + (\mathbf{u} \wedge \mathbf{w})$$

for every scalar  $\alpha$ , and vectors  $\mathbf{u}, \mathbf{v}$ , and  $\mathbf{w}$  in  $\mathbb{E}_2$ .

In order to investigate the nature of the multivector  $\mathbf{u} \wedge \mathbf{v}$ , let us observe that, if  $\{\mathbf{g}_1, \mathbf{g}_2\}$  is an orthogonal basis for  $\mathbb{E}_2$ ,  $\mathbf{u} = \mu_1 \mathbf{g}_1 + \mu_2 \mathbf{g}_2$ , and  $\mathbf{v} = \nu_1 \mathbf{g}_1 + \nu_2 \mathbf{g}_2$ , then

$$\mathbf{g}_1 \cdot \mathbf{g}_2 = 0$$
,  $\mathbf{g}_1 \wedge \mathbf{g}_2 = \mathbf{g}_1 \mathbf{g}_2$ , and  $\mathbf{u} \wedge \mathbf{v} = \det \begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} \mathbf{g}_1 \mathbf{g}_2$  (6)

We now prove that the dimension of the associative vector algebra  $\mathbb{G}_2$  is 4.

**Proposition 1.** If  $\{\mathbf{g}_1, \mathbf{g}_2\}$  is an orthogonal basis of  $\mathbb{E}_2$ , then  $\{1, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_1\mathbf{g}_2\}$  is a basis of  $\mathbb{G}_2$ .

*Proof.* Let us prove that  $\{1, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_1\mathbf{g}_2\}$  generates  $\mathbb{G}_2$ .

Every element in  $\mathbb{G}_2$  is a polynomial of vectors, that is a linear combination of product of vectors. As every vector is a linear combination of vectors  $\mathbf{g}_1$ , and  $\mathbf{g}_2$ , by the distributivity property, every product of vector can be written as a linear combination of product of vectors  $\mathbf{g}_1$ , and  $\mathbf{g}_2$ . As vectors  $\mathbf{g}_1$ , and  $\mathbf{g}_2$  anticommute  $(\mathbf{g}_1\mathbf{g}_2 = -\mathbf{g}_2\mathbf{g}_1)$ , every product of vectors  $\mathbf{g}_1$ , and  $\mathbf{g}_2$  can be reduced to the product  $\epsilon(\mathbf{g}_1)^h(\mathbf{g}_2)^k$  for some integer exponents  $h, k \in \mathbb{N}$  and  $\epsilon \in \{1, -1\}$ . As  $(\mathbf{g}_i)^l$  is a scalar if the integer exponent l is even, and it is a multiple of vector  $\mathbf{g}_i$  if l is odd, then every poly-vector can be rewritten as a linear combination of elements in  $\{1, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_1\mathbf{g}_2\}$ . This proves that  $\{1, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_1\mathbf{g}_2\}$  generates  $\mathbb{G}_2$ . It remains to show that  $1, \mathbf{g}_1, \mathbf{g}_2$ , and  $\mathbf{g}_1\mathbf{g}_2$  are linearly independent. This means that we have to show that, if

$$\alpha_0 + \alpha_1 \mathbf{g}_1 + \alpha_2 \mathbf{g}_2 + \alpha_3 \mathbf{g}_1 \mathbf{g}_2 = 0 \tag{7}$$

then necessarily  $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ . First of all, let us observe that  $\mathbf{g}_1\mathbf{g}_2 \neq 0$ . By contradiction, if  $\mathbf{g}_1\mathbf{g}_2 = 0$ , then by multiplying it from left by  $(\mathbf{g}_1)^{-1}$  (recall that both vectors  $\mathbf{g}_1$ , and  $\mathbf{g}_2$  are invertible), we would obtain  $\mathbf{g}_2 = 0$ , a contradiction with axiom A5. Let us now multiply the expression (7) from left by  $\mathbf{g}_1$ , and from right by  $(\mathbf{g}_1)^{-1}$ . Then, we can write the following equivalent relations

$$\mathbf{g}_{1}(\alpha_{0} + \alpha_{1}\mathbf{g}_{1} + \alpha_{2}\mathbf{g}_{2} + \alpha_{3}\mathbf{g}_{1}\mathbf{g}_{2})(\mathbf{g}_{1})^{-1} = 0$$

$$\mathbf{g}_{1}\alpha_{0}(\mathbf{g}_{1})^{-1} + \mathbf{g}_{1}\alpha_{1}\mathbf{g}_{1}(\mathbf{g}_{1})^{-1} + \mathbf{g}_{1}\alpha_{2}\mathbf{g}_{2}(\mathbf{g}_{1})^{-1} + \mathbf{g}_{1}\alpha_{3}\mathbf{g}_{1}\mathbf{g}_{2}(\mathbf{g}_{1})^{-1} = 0$$

$$\alpha_{0} + \alpha_{1}\mathbf{g}_{1} + \alpha_{2}\mathbf{g}_{1}\mathbf{g}_{2}(\mathbf{g}_{1})^{-1} + \alpha_{3}\mathbf{g}_{1}\mathbf{g}_{1}\mathbf{g}_{2}(\mathbf{g}_{1})^{-1} = 0$$

$$\alpha_{0} + \alpha_{1}\mathbf{g}_{1} - \alpha_{2}\mathbf{g}_{2}\mathbf{g}_{1}(\mathbf{g}_{1})^{-1} - \alpha_{3}\mathbf{g}_{1}\mathbf{g}_{2}\mathbf{g}_{1}(\mathbf{g}_{1})^{-1} = 0$$

$$\alpha_{0} + \alpha_{1}\mathbf{g}_{1} - \alpha_{2}\mathbf{g}_{2} - \alpha_{3}\mathbf{g}_{1}\mathbf{g}_{2} = 0$$

By summing the last relation with (7), and then dividing the result by two, we obtain that

$$\alpha_0 + \alpha_1 \mathbf{g}_1 = 0$$

Rule A5 implies that  $\alpha_0 = \alpha_1 = 0$ . So, the initial expression (7) reduces to

$$\alpha_2 \mathbf{g}_2 + \alpha_3 \mathbf{g}_1 \mathbf{g}_2 = 0 \tag{8}$$

Let us multiply the foregoing expression from left by  $\mathbf{g}_2$ , and from right by  $(\mathbf{g}_2)^{-1}$ . Then, we can write the following equivalent relations

$$\mathbf{g}_2 \alpha_2 \mathbf{g}_2 (\mathbf{g}_2)^{-1} + \mathbf{g}_2 \alpha_3 \mathbf{g}_1 \mathbf{g}_2 (\mathbf{g}_2)^{-1} = 0$$
  
$$\alpha_2 \mathbf{g}_2 + \alpha_3 \mathbf{g}_2 \mathbf{g}_1 = 0$$
  
$$\alpha_2 \mathbf{g}_2 - \alpha_3 \mathbf{g}_1 \mathbf{g}_2 = 0$$

As before, by summing the last relation with (8), and then dividing the result by two, we obtain that  $\alpha_2 = 0$ . So, we end to the expression  $\alpha_3 \mathbf{gg}_2 = 0$ . By multiplying it from left by  $(\mathbf{g}_2)^{-1}(\mathbf{g}_1)^{-1}$ , we obtain  $\alpha_3 = 0$ .

The foregoing result and (6) imply that  $\mathbf{u} \wedge \mathbf{v} = 0$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are linearly dependent. We can also notice that the geometric product of two orthonormal vectors does not depend on those particular factors, but only on their order.

**Proposition.** If the basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$  for  $\mathbb{E}_2$  is not only orthogonal, but also orthonormal (that is,  $(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = 1$ ), then the geometric product  $\mathbf{e}_1\mathbf{e}_2 \in \mathbb{G}_2$  does not depend on the particular orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$ , but only on its orientation (that is, on the order the two vectors of the basis are given).

*Proof.* If  $\{\mathbf{e}'_1, \mathbf{e}'_2\}$  is any other orthonormal basis for  $\mathbb{E}_2$ ,  $\mathbf{e}'_1 = \epsilon_{1,1}\mathbf{e}_1 + \epsilon_{1,2}\mathbf{e}_2$  and  $\mathbf{e}'_2 = \epsilon_{2,1}\mathbf{e}_1 + \epsilon_{2,2}\mathbf{e}_2$ , then

$$\mathbf{e}_1'\mathbf{e}_2' = \det \left(\begin{array}{cc} \epsilon_{1,1} & \epsilon_{1,2} \\ \epsilon_{2,1} & \epsilon_{2,2} \end{array}\right) \mathbf{e}_1 \mathbf{e}_2 = \pm \mathbf{e}_1 \mathbf{e}_2$$

as 
$$\begin{pmatrix} \epsilon_{1,1} & \epsilon_{1,2} \\ \epsilon_{2,1} & \epsilon_{2,2} \end{pmatrix}$$
 is an orthogonal matrix.

That is why we can denote the multivector  $\mathbf{e}_1\mathbf{e}_2 = I_2$ , and call it an *orientation* of  $\mathbb{E}_2$ . So, by Proposition 1, the basic multivectors in  $\mathbb{G}_2$  are scalars, vectors, and multiples of orientations. You can also notice that

$$(I_2)^2 = I_2 I_2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_1 (\mathbf{e}_2 \mathbf{e}_1) \mathbf{e}_2 = \mathbf{e}_1 (-\mathbf{e}_1 \mathbf{e}_2) \mathbf{e}_2 = -\mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 = -(\mathbf{e}_1 \mathbf{e}_1)(\mathbf{e}_2 \mathbf{e}_2) = -1$$

So, we have that  $I_2$  is invertible in  $\mathbb{G}_2$  and  $(I_2)^{-1} = -I_2$ . Doesn't  $I_2$  strongly resemble the imaginary unit ?

Finally, we can write the geometric product between two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{E}_2$ in terms of their coordinates with respect to any orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$  of  $\mathbb{E}_2$ ,

$$\mathbf{u}\mathbf{v} = (\mathbf{u} \cdot \mathbf{v}) + \mathbf{u} \wedge \mathbf{v} = (\mu_1 \nu_1 + \mu_2 \nu_2) + \det \begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} I_2$$
$$= |\mathbf{u}| |\mathbf{v}|(\cos \theta + I_2 \sin \theta)$$

where  $\mathbf{u} = \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e} - 2$ , and  $\mathbf{v} = \nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2$ , angle  $\theta$  is oriented in  $\mathbb{E}_2$  so that vector  $\mathbf{u}$  can rotate towards vector  $\mathbf{v}$  spanning angle  $\theta \in (0, \pi)$ , provided  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent (otherwise,  $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v}$ ).

#### 2.3 Comparing vectors in $\mathbb{E}_2 \subset \mathbb{G}_2$ with complex numbers.

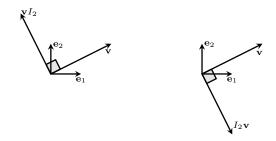
You have probably remarked some similarities with complex numbers ( $I_2$  algebraically behaves in  $\mathbb{G}_2$  as the imaginary unit *i* does in  $\mathbb{C}$ ). However, you should notice that while  $\mathbb{G}_2$  has real-dimension four,  $\mathbb{C}$  has real-dimension two. That is, scalars, vectors and orientations are distinguished in  $\mathbb{G}_2$ , while in  $\mathbb{C}$  they all collapse into the notion of "complex number", viewed as a two-dimensional real vector. In other words, the geometric product of two vectors in  $\mathbb{E}_2$  is not itself a vector: it is the sum of a scalar and a multiple of the orientation  $I_2$ . Instead, the product of two complex numbers (viewed as vectors in  $\mathbb{R}^2$ ) is still a complex number (i.e., a vector in  $\mathbb{R}^2$ ). Moreover,  $\mathbb{C}$  is a division algebra (every non-zero element is invertible), while  $\mathbb{G}_2$  is not:  $1 + \mathbf{e}_1$ , and  $1 - \mathbf{e}_1$  are not invertible in  $\mathbb{G}_2$ , as  $(1 + \mathbf{e}_1)(1 - \mathbf{e}_1) = 0$ . The product in  $\mathbb{C}$  is commutative, while the geometric product in  $\mathbb{G}_2$  is not. The geometric product in  $\mathbb{G}_2$  is invariant by rotations in  $\mathbb{E}_2$ , while the product in  $\mathbb{C}$  is not invariant by rotations in  $\mathbb{R}^2$ .

#### **2.4** The determinant of a $2 \times 2$ real matrix viewed in $\mathbb{G}_2$ .

Let us notice that, for every vector  $\mathbf{v} \in \mathbb{E}_2$ , then

$$\mathbf{v}I_2 = (\nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2)\mathbf{e}_1\mathbf{e}_2 = \nu_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \nu_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\nu_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 - \nu_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2$$
  
=  $-\mathbf{e}_1\mathbf{e}_2(\nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2) = -I_2\mathbf{v} = \nu_1\mathbf{e}_2 - \nu_2\mathbf{e}_1 \in \mathbb{E}_2$ 

which corresponds to the vector obtained by rotating vector  $\mathbf{v}$  of a right angle according to the orientation  $I_2 = \mathbf{e}_1 \mathbf{e}_2$ , as illustrated below,



Thanks to the foregoing results we can now proceed to show the two key facts that link the geometric product to the basic notion of secant plane.

**Proposition.** The determinant of a 2x2 real matrix is a Clifford ratio in  $\mathbb{G}_2$ .

*Proof.* Let us consider the rows of a  $2 \times 2$  real matrix

$$\left(\begin{array}{cc} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{array}\right)$$

as the components of two vectors  $\mathbf{u} = \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e}_2$ , and  $\mathbf{v} = \nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2$  in  $\mathbb{E}_2$  with respect to some orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$ .

As 
$$\mathbf{u} \wedge \mathbf{v} = \det \begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} \mathbf{e}_1 \mathbf{e}_2 = \det \begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} I_2$$
, then we can write  
$$\det \begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} = (\mathbf{u} \wedge \mathbf{v})(I_2)^{-1}$$
, which is a ratio in  $\mathbb{G}_2$ .

**Remark.** You can verify that  $(\mathbf{u} \wedge \mathbf{v})(I_2)^{-1} = (I_2)^{-1}(\mathbf{u} \wedge \mathbf{v})$ . So the expression det  $\begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} = \frac{\mathbf{u} \wedge \mathbf{v}}{I_2}$  for the determinant of a 2 × 2 matrix is unambiguous.

**Proposition 2.** The determinant a 2x2 real matrix is a scalar product.

*Proof.* By using the same assumptions of the foregoing proof, we have that

$$\det \begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} = (\mathbf{u} \wedge \mathbf{v})(I_2)^{-1} = -(\mathbf{u} \wedge \mathbf{v})I_2 = (\mathbf{v} \wedge \mathbf{u})I_2 = \frac{1}{2}(\mathbf{v}\mathbf{u} - \mathbf{u}\mathbf{v})I_2$$
$$= \frac{1}{2}(\mathbf{v}\mathbf{u}I_2 - \mathbf{u}\mathbf{v}I_2) = \frac{1}{2}[\mathbf{v}(\mathbf{u}I_2) + (\mathbf{u}I_2)\mathbf{u}] = (\mathbf{u}I_2) \cdot \mathbf{v} .$$

# 3 The difference vector quotient of a secant plane.

Let us reformulate the relations defining the plane secant the graph of a function. In the one-variable case the equation of the line secant the graph of  $g: I \to \mathbb{R}$  at

points  $(a, g(a)), (b, g(b)) \in I \times \mathbb{R}$  can be written, in the Cartesian (x, z)-plane, in two equivalent ways: (3) or

$$\det \left(\begin{array}{cc} x-a & z-g(a) \\ b-a & g(b)-g(a) \end{array}\right) = 0 \ .$$

In the (x, y, z)-coordinate system, the equation of the plane secant the graph of f at points  $(\alpha_1, \alpha_2, f(\mathbf{a})), (\beta_1, \beta_2, f(\mathbf{b})), (\gamma_1, \gamma_2, f(\mathbf{c})) \in \mathbb{R}^3$  can be written as

$$\det \begin{pmatrix} x - \alpha_1 & y - \alpha_2 & z - f(\mathbf{a}) \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & f(\mathbf{b}) - f(\mathbf{a}) \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 & f(\mathbf{c}) - f(\mathbf{a}) \end{pmatrix} = 0$$
(9)

where  $\mathbf{a} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ ,  $\mathbf{b} = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2$ ,  $\mathbf{c} = \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2$  are vectors (that we also call points) in the two-dimensional domain  $\Omega \subseteq \mathbb{E}_2$  of f, and  $\{\mathbf{e}_1, \mathbf{e}_2\}$  is an orthonormal basis of  $\mathbb{E}_2$ . In the following, we show how to rewrite (9) in the  $\mathbb{E}_2$ -coordinate-free setting (4). The following equivalences start from a Laplace expansion of the determinant (9).

$$(z - f(\mathbf{a})) \det \left( \begin{array}{cc} \beta_1 - \alpha_1 & \beta_2 - \alpha_2 \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 \end{array} \right) - (f(\mathbf{b}) - f(\mathbf{a})) \det \left( \begin{array}{cc} x - \alpha_1 & y - \alpha_2 \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 \end{array} \right) + \\ + (f(\mathbf{c}) - f(\mathbf{a})) \det \left( \begin{array}{cc} x - \alpha_1 & y - \alpha_2 \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 \end{array} \right) = 0 \\ (z - f(\mathbf{a})) \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right] (I_2)^{-1} = \\ (f(\mathbf{b}) - f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right] (I_2)^{-1} - (f(\mathbf{c}) - f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] (I_2)^{-1} \\ (z - f(\mathbf{a})) \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right] = \\ (f(\mathbf{b}) - f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right] - (f(\mathbf{c}) - f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \\ z - f(\mathbf{a}) = (f(\mathbf{b}) - f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + -(f(\mathbf{c}) - \\ f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{c}) - \\ f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{c}) - \\ f(\mathbf{a})) \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{c}) - \\ f(\mathbf{a})) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{c}) - \\ f(\mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{c}) - \\ f(\mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{c}) - \\ f(\mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a}) - \\ f(\mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a}) - \\ f(\mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a}) - \\ f(\mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a}) - \\ f(\mathbf{a} - \mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \wedge (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a}) - \\ f(\mathbf{a} - \mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a} - \mathbf{a}) - \\ f(\mathbf{a} - \mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a} - \mathbf{a}) - \\ f(\mathbf{a} - \mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a} - \mathbf{a}) - \\ f(\mathbf{a} - \mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf{a} - \mathbf{a}) - \\ f(\mathbf{a} - \mathbf{a}) \left[ (\mathbf{a} - \mathbf{a}) \right]^{-1} + (f(\mathbf$$

$$z = f(\mathbf{a}) + (f(\mathbf{b}) - f(\mathbf{a})) [(\mathbf{x} - \mathbf{a}) \land (\mathbf{c} - \mathbf{a})] [(\mathbf{b} - \mathbf{a}) \land (\mathbf{c} - \mathbf{a})]^{-1} + -(f(\mathbf{c}) - f(\mathbf{a})) [(\mathbf{x} - \mathbf{a}) \land (\mathbf{b} - \mathbf{a})] [(\mathbf{b} - \mathbf{a}) \land (\mathbf{c} - \mathbf{a})]^{-1}$$

As 
$$(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a}$$
  
=  $I_2 \underbrace{\left[ \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} + \det \begin{pmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} + \det \begin{pmatrix} \gamma_1 & \gamma_2 \\ \alpha_1 & \alpha_2 \end{pmatrix} \right]}_{2\tau}$ ,

where  $\tau$  is simply the oriented<sup>11</sup> area of the triangle having as vertices the points **a**, **b**, and **c**), the foregoing equivalences, describing the secant plane, can continue as follows

<sup>&</sup>lt;sup>11</sup>The sign of  $\tau$  is positive if and only if the geometric ratio between  $(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})$  and  $I_2$  is positive. See also [1].

$$z = f(\mathbf{a}) + \frac{f(\mathbf{b}) - f(\mathbf{a})}{2\tau} \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right] I_2^{-1} - \frac{f(\mathbf{c}) - f(\mathbf{a})}{2\tau} \left[ (\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) \right] I_2^{-1}$$
$$z = f(\mathbf{a}) - \frac{f(\mathbf{b}) - f(\mathbf{a})}{2\tau} \left[ (\mathbf{c} - \mathbf{a}) \wedge (\mathbf{x} - \mathbf{a}) \right] I_2^{-1} + \frac{f(\mathbf{c}) - f(\mathbf{a})}{2\tau} \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{x} - \mathbf{a}) \right] I_2^{-1}$$

By proposition 2, we can write

$$z = f(\mathbf{a}) - \frac{f(\mathbf{b}) - f(\mathbf{a})}{2\tau} \left\{ \left[ (\mathbf{c} - \mathbf{a})I_2 \right] \cdot (\mathbf{x} - \mathbf{a}) \right\} + \frac{f(\mathbf{c}) - f(\mathbf{a})}{2\tau} \left\{ \left[ (\mathbf{b} - \mathbf{a})I_2 \right] \cdot (\mathbf{x} - \mathbf{a}) \right\} \right\}$$
$$z = f(\mathbf{a}) - \left\{ \frac{f(\mathbf{b}) - f(\mathbf{a})}{2\tau} \left[ (\mathbf{c} - \mathbf{a})I_2 \right] - \frac{f(\mathbf{c}) - f(\mathbf{a})}{2\tau} \left[ (\mathbf{b} - \mathbf{a})I_2 \right] \right\} \cdot (\mathbf{x} - \mathbf{a})$$
$$z = f(\mathbf{a}) - \frac{1}{2\tau} \left\{ \left[ \left( f(\mathbf{b}) - f(\mathbf{a}) \right) (\mathbf{c} - \mathbf{a}) - \left( f(\mathbf{c}) - f(\mathbf{a}) \right) (\mathbf{b} - \mathbf{a}) \right] I_2 \right\} \cdot (\mathbf{x} - \mathbf{a})$$

This allows to explicitly write vector  $\mathbf{q} = \mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$  of expression (4)

$$\mathbf{q} = -\frac{1}{2\tau} \left\{ \left[ \left( f(\mathbf{b}) - f(\mathbf{a}) \right) (\mathbf{c} - \mathbf{a}) - \left( f(\mathbf{c}) - f(\mathbf{a}) \right) (\mathbf{b} - \mathbf{a}) \right] I_2 \right\}$$
$$= \left[ \left( f(\mathbf{b}) - f(\mathbf{a}) \right) (\mathbf{c} - \mathbf{a}) - \left( f(\mathbf{c}) - f(\mathbf{a}) \right) (\mathbf{b} - \mathbf{a}) \right] \left[ (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \right]^{-1},$$

and proves the following main result of this article.

**Theorem 1.** The plane secant the graph of a two-variable function  $f : \Omega \subseteq \mathbb{E}_2 \to \mathbb{R}$  at points  $(\mathbf{a}, f(\mathbf{a}))$ ,  $(\mathbf{b}, f(\mathbf{b}))$ , and  $(\mathbf{c}, f(\mathbf{c}))$  (where  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are three non-collinear points in the domain  $\Omega$  of f), can be represented in the  $(\mathbf{v}, z)$ -space  $\mathbb{E}_2 \times \mathbb{R}$  by the relation

$$z = f(\mathbf{a}) + \mathbf{q} \cdot (\mathbf{v} - \mathbf{a})$$
,

where vector  $\mathbf{q} = \mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$  is the geometric quotient in  $\mathbb{G}_2$  between the vector

$$(f(\mathbf{b}) - f(\mathbf{a}))(\mathbf{c} - \mathbf{a}) - (f(\mathbf{c}) - f(\mathbf{a}))(\mathbf{b} - \mathbf{a}),$$

and the orientation

$$(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a}).$$

# 4 The vector quotient as a linear combination in $\mathbb{E}_2$ .

Computations in  $\mathbb{G}_2$  allow to explicitly express the foregoing difference vector quotient  $\mathbf{q} = \mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$  as a linear combination of normals to the triangle defined by  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

Let  $\partial_{\mathbf{a}} = \mathbf{c} - \mathbf{b}$ ,  $\partial_{\mathbf{b}} = \mathbf{a} - \mathbf{c}$ ,  $\partial_{\mathbf{c}} = \mathbf{b} - \mathbf{a}$ ; so,  $\partial_{\mathbf{a}} + \partial_{\mathbf{b}} + \partial_{\mathbf{c}} = 0$ . Moreover

$$(\mathbf{b}-\mathbf{a}) \wedge (\mathbf{c}-\mathbf{a}) = \partial_{\mathbf{a}} \wedge \partial_{\mathbf{b}} = \partial_{\mathbf{b}} \wedge \partial_{\mathbf{c}} = \partial_{\mathbf{c}} \wedge \partial_{\mathbf{a}} = 2\tau I_2,$$

where  $\tau$  is the signed area of the oriented triangle determined by the ordered points **a**, **b**, and **c**, whose sign depends on the orientation  $I_2$ . Then,

$$\begin{split} & \left(f(\mathbf{b}) - f(\mathbf{a})\right)(\mathbf{c} - \mathbf{a}) - \left(f(\mathbf{c}) - f(\mathbf{a})\right)(\mathbf{b} - \mathbf{a}) = \left(f(\mathbf{a}) - f(\mathbf{b})\right)\partial_{\mathbf{b}} - \left(f(\mathbf{c}) - f(\mathbf{a})\right)\partial_{\mathbf{c}} \\ &= f(\mathbf{a})\left[\partial_{\mathbf{b}} + \partial_{\mathbf{c}}\right] - f(\mathbf{b})\partial_{\mathbf{b}} - f(\mathbf{c})\partial_{\mathbf{c}} = -\left[f(\mathbf{a})\partial_{\mathbf{a}} + f(\mathbf{b})\partial_{\mathbf{b}} + f(\mathbf{c})\partial_{\mathbf{c}}\right] \\ &= f(\mathbf{a})\left(\partial_{\mathbf{b}} + \partial_{\mathbf{c}}\right) + f(\mathbf{b})\left(\partial_{\mathbf{a}} + \partial_{\mathbf{c}}\right) + f(\mathbf{c})\left(\partial_{\mathbf{a}} + \partial_{\mathbf{b}}\right) \\ &= \left[f(\mathbf{a}) + f(\mathbf{b})\right]\partial_{\mathbf{c}} + \left[f(\mathbf{a}) + f(\mathbf{c})\right]\partial_{\mathbf{b}} + \left[f(\mathbf{b}) + f(\mathbf{cb})\right]\partial_{\mathbf{a}} \end{split}$$

So, we can write

$$\mathbf{q} = \left\{ \left[ f(\mathbf{a}) + f(\mathbf{b}) \right] \partial_{\mathbf{c}} + \left[ f(\mathbf{a}) + f(\mathbf{c}) \right] \partial_{\mathbf{b}} + \left[ f(\mathbf{b}) + f(\mathbf{c}) \right] \partial_{\mathbf{a}} \right\} \left( 2\tau I_2 \right)^{-1} \\ = \frac{f(\mathbf{a}) + f(\mathbf{b})}{2\tau} \partial_{\mathbf{c}} (I_2)^{-1} + \frac{f(\mathbf{a}) + f(\mathbf{c})}{2\tau} \partial_{\mathbf{b}} (I_2)^{-1} + \frac{f(\mathbf{b}) + f(\mathbf{c})}{2\tau} \partial_{\mathbf{a}} (I_2)^{-1} \\ = \frac{f(\mathbf{a}) + f(\mathbf{b})}{2\tau} \underbrace{I_2 \partial_{\mathbf{c}}}_{\partial_{\mathbf{c}}^{\perp}} + \frac{f(\mathbf{a}) + f(\mathbf{c})}{2\tau} \underbrace{I_2 \partial_{\mathbf{b}}}_{\partial_{\mathbf{b}}^{\perp}} + \frac{f(\mathbf{b}) + f(\mathbf{c})}{2\tau} \underbrace{I_2 \partial_{\mathbf{a}}}_{\partial_{\mathbf{a}}^{\perp}} .$$

This proves our second theorem, which generalizes a lemma proved<sup>12</sup> in [4].

**Theorem 2.** The plane secant the graph of a two-variable function  $f : \Omega \subseteq \mathbb{E}_2 \to \mathbb{R}$  at points  $(\mathbf{a}, f(\mathbf{a}))$ ,  $(\mathbf{b}, f(\mathbf{b}))$ , and  $(\mathbf{c}, f(\mathbf{c}))$  (where  $\mathbf{a}, \mathbf{b}$ , and  $\mathbf{c}$  are three non-collinear points in the domain  $\Omega$  of f), can be represented in the  $(\mathbf{v}, z)$ -space  $\mathbb{E}_2 \times \mathbb{R}$  by the relation  $z = f(\mathbf{a}) + \mathbf{q} \cdot (\mathbf{v} - \mathbf{a})$ , where vector  $\mathbf{q} = \mathbf{q}_{(f, \mathbf{a}, \mathbf{b}, \mathbf{c})}$  is the linear combination (5).

**Remark.** If f is the affine function  $f(\mathbf{x}) = (\mathbf{v} \cdot \mathbf{x}) + \phi$ , for some  $\mathbf{v} \in \mathbb{E}_2$  and  $\phi \in \mathbb{R}$ , then you can verify that  $\mathbf{q}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \mathbf{v}$ .

## References

- [1] B. Braden, The Surveyor's Area Formula, Coll. Math. J. 17 (1986), 326–337.
- [2] B.M. Budak, S.V. Fomin, Multiple Integrals, Field Theory and Series, MIR Publishers, Moscow, 1973.
- [3] Y. Choquet-Bruhat, C. DeWitt-Morette, Analysis, manifolds, and physics Part I: Basics, North Holland, 1982.
- [4] J. D. Jr. Daws, A Fundamental Theorem of Multivariable Calculus, Chancellor's Honors Program Projects, (2013). https://trace.tennessee.edu/utk\_chanhonoproj/1604
- [5] L. Dorst, D. Fontijne, S. Mann, Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry, The Morgan Kaufmann Series in Computer Graphics, Elsevier, 2007.

<sup>&</sup>lt;sup>12</sup>Only for linear functions.

- [6] M. Esser, O. Shisha, A modified differentiation, Amer. Math.Monthly 71 (1964), 904–906.
- [7] K. Guerlebeck, W. Sproessig, Quaternionic and Clifford calculus for physicists and engineers, Wiley, 1997.
- [8] A. Macdonald, *Linear and Geometric Algebra*, CreateSpace Independent Publishing Platform, Lexington, KY, 2011 and 2022.
- [9] G. Peano, Sulla definizione dell'area d'una superficie, Atti della Reale Accademia dei Lincei: Rendiconti 4 (1890), 54–57.
- [10] G. Peano, Sur la définition de la dérivée, Mathesis Recueil mathématique à l'usage des écoles spéciales et des établissements d'instruction moyenne 2 (1892), 12–14.
- [11] I.R. Porteous, *Topological Geometry*, Second edition, Cambridge University Press, 1981.
- [12] M. Riesz, *Clifford numbers and spinors*, Kluwer Academic Publishers, 1993.
- [13] H.A. Schwarz, Sur une définition erronée de l'aire, d'une surface courbe, Gesammelte Mathematische Abhandlungen von H. A. Schwarz, Springer, (1890), 309-311. https://archive.org/details/gesammeltemathem02schwuoft/page/308/mode/2up
- [14] J. Vaz, R. d. Rocha, An introduction to Clifford algebras and spinors, Oxford university press, 2016.
- [15] P. Zames, Surface Area and the Cylinder Area Paradox, Coll. Math. J. 8 (1977), 207–211. (This article obtained in 1978 the George Polya Award)